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are relatively prime. In the case above $a - b = \pm 1$, whence either (1) $(m - n)^2 - 2n^2 = 1$, or (2) $(m + n)^2 - 2m^2 = 1$. The equation (2) may be derived from (1) by interchanging m and n and changing the sign of n. This would not affect the value of c, nor the algebraic work below,[1] hence we need consider only (1).

Writing (1) in the form $(m - n)^2 - 1 = 2n^2$, and factoring we have

$$[m - n - 1][m - n + 1] = 2n^2. \tag{3}$$

Since the two factors differ by 2 and their product is even, each is even, 2 is the H.C.F., and one factor is twice an odd square and the other the square of an even number. Accordingly we have

$$(4)\ \begin{aligned} m - n - 1 &= 2\alpha^2 \\ m - n + 1 &= 4\beta^2 \end{aligned} \qquad \text{or} \qquad (5)\ \begin{aligned} m - n - 1 &= 4\beta^2 \\ m - n + 1 &= 2\alpha^2, \end{aligned}$$

where α and β denote integers. From (3), (4) and (5) we find

$$n = 2\alpha\beta, \qquad m = \alpha^2 + 2\beta^2 + 2\alpha\beta, \qquad \pm 1 = 2\beta^2 - \alpha^2, \tag{6}$$

the upper sign resulting from equations (4): the lower, from (5). Substituting these values in the expression for c, we find

$$c = m^2 + n^2 = (\alpha^2 + 2\beta^2 + 2\alpha\beta)^2 + (2\alpha\beta)^2 = (\alpha^2 + 2\alpha\beta)^2 + (2\alpha\beta + 2\beta^2)^2 + (2\alpha\beta)^2.$$

But $(\alpha^2 + 2\alpha\beta) - (2\alpha\beta + 2\beta^2) = \mp 1$ from (6), so that c is expressed in the desired form.

It is worthy of note that every solution of the equation $\lambda^2 - 2\mu^2 = \pm 1$ leads to a triangle having the given property, and such triangles can be obtained only from such solutions. All these can be found by developing $\sqrt{2}$ into a continued fraction and taking the numerator and denominator of any convergent as λ and μ respectively. In this way we get the following set of solutions:

λ	μ	m	n	a	b	c
1	1	2	1	3	4	$5 = 0^2 + 1^2 + 2^2$
		5	2	21	20	$29 = 2^2 + 3^2 + 4^2$
3	2	12	5	119	120	$169 = 3^2 + 4^2 + 12^2$
		29	12	697	696	$985 = 12^2 + 20^2 + 21^2$
7	5	70	29	4059	4060	$5741 = 20^2 + 21^2 + 70^2$.

3]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given a parallelogram with center O, vertices $PQP'Q'$, mid-points of sides $ABA'B'$ (cyclic order $PAQBP'A'Q'B'$). Let K be any point of OA. Draw KLH parallel to $Q'Q$ cutting AQ at L, and draw BLM, meeting OA produced at M. Draw MH, parallel to PP', to meet KLH at H, and draw $B'KE$ to meet BL at E. Repeat, changing A, B, P, Q to A', B', P', Q', respectively, and *vice versa*. Repeat each of the foregoing, changing P, P', B, B' to Q, Q', B', B, respectively, and *vice versa*.

What are the loci of E and H? Show that EH passes through A' when K is a point of OA, through A when K is a point of OA'. Consider the effect of interchanging the rôles of A and B.

(This construction, as commonly given, is specialized in these particulars: the parallelogram is rectangular, the divisions of OA are equal, and the locus of H is not found.)

I. Solution by Arthur Pelletier, Montreal, Can.

Let $OB = a$ and $OA' = b$ be the axes of coördinates and, taking $OK = -mb$, we find the following equations of the indicated straight lines:

(1) (KLH) $x/a + y/b = -m$,
(2) (BLM) $x/a - my/b = 1$,
(3) (MH) $x/a - y/b = 1/m$,
(4) ($B'KE$) $x/a + y/mb = -1$.

By eliminating m from (2) and (4) we find for the equation of the locus of E, $x^2/a^2 + y^2/b^2 = 1$. Hence when K passes from A to O the point E describes the part AB of an ellipse inscribed in the parallelogram, tangent to the sides at A, B, A', B'. The other parts of the ellipse will evidently be obtained by making the changes indicated in the problem. Similarly, we obtain from (1) and (3) the equation $x^2/a^2 - y^2/b^2 = -1$. Hence the locus of H is the part AHC of an hyperbola tangent to PQ at A etc.

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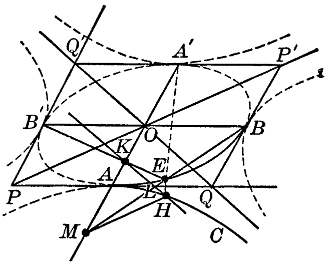
[1] The new n would be negative, consequently $\alpha\beta$ in (6) would be negative. Two of the three numbers, the sum of whose squares is c, would be $2\alpha\beta - \alpha^2$, $2\alpha\beta - 2\beta^2$ and the difference is again ± 1.

If we add to the two sides of (1) the corresponding sides of (3) multiplied by m, we obtain $(1+m)(x/a)+(1-m)y/b=1-m$, the equation of a straight line through H. The same process applied to (2) and (4) yields the *same* equation and, since the point $(0, b)$ satisfies the new equation, we see that H, E, and A' lie on this straight line.

Note.—A geometrical solution may be given. Starting with a square we easily find as loci, a circle and two conjugate equilateral hyperbolas. Then, by projection, we derive the loci found above. By the theorem of Meneläus we prove that the correlative points E, H, and A' are collinear, a property preserved in projection.

II. Solution by Otto Dunkel, Washington University.



The ranges of points K and L are projective and hence E is the intersection of two projective pencils with centers at B' and B, such that $B'B$ is not self-corresponding. It thus follows that the locus of E is a conic tangent at B' and B and passing through A and A' of the sides of the parallelogram. By using A and A' as centers it will be seen that it is tangent to the other two sides PQ and $P'Q'$. Similarly, the range of points M is projective with the ranges L and K, and hence H is the intersection of two pencils with centers at the points at infinity on OQ and OP. Thus the locus of H is a conic with OQ and OP as asymptotes, and the conic passes through A and A'. It may be shown by the theory of conics (analytic or projective theory) that PQ is a tangent. It also follows that the pencils $A'(E)$ and $A'(H)$ are projective, that $A'A$, $A'B$, $A'B'$ are self-corresponding rays of these two pencils and hence all the corresponding rays coincide. Therefore, A', E, H lie on a straight line, etc.

2757 [1919, 124]. Proposed by E. P. LANE, Rice Institute, Houston, Texas.

Integrate by quadrature the differential equation

$$\frac{d^2y}{dx^2}-3y\frac{dy}{dx}+y^3=0.$$

Solution by Alexander Dillingham, U. S. Naval Academy.

Interchanging the variables and setting $q=dx/dy$, we have $d^2y/dx^2=-(dq/dy)/q^3$. The given equation becomes, after making these substitutions,

$$\frac{dq}{dy}+3yq^2-y^3q^3=0.$$

By inspection we find a particular integral $q_1=y^{-2}$ and hence we are led to put $q=q_1+v$, where v is a function of y. The equation (1) now becomes a Bernoulli equation $dv/dy+3v/y=v^3y^3$ which reduces, on putting $z=v^{-2}$, to the linear equation $dz/dy-6z/y=-2y^3$ with the integrating factor y^{-6}. We then obtain $zy^{-6}=\int(-2y^{-3})dy=y^{-2}+c_1$ or $z=y^4+c_1y^6=v^{-2}$. Hence we have in turn

$$v=\pm\frac{1}{y^2\sqrt{1+c_1y^2}},\qquad q=\frac{1}{y^2}\pm\frac{1}{y^2\sqrt{1+c_1y^2}}=\frac{dx}{dy},$$

and by integrating the last equation, we have finally

$$x=-\frac{1}{y}\mp\sqrt{y^{-2}+c_1}+c_2.$$

Also solved by R. D. Bohannan, P. J. da Cunha, E. B. Escott, M. Gutten, H. Halperin, H. L. Olson, and Elijah Swift.